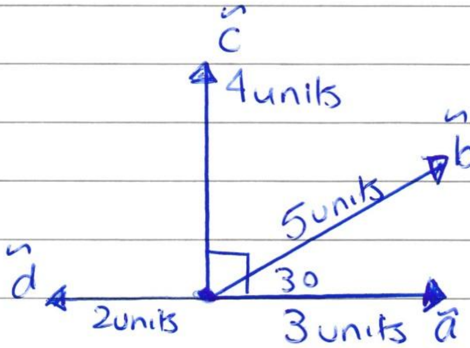
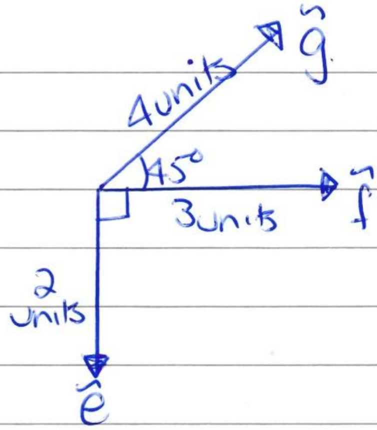
Specialist Mathematics Unit 1: **Chapter 8**

Ex 8A.





1. $\tilde{a} \cdot \tilde{b} = 3 \times 5 \times \cos 30^\circ$
$= \frac{15\sqrt{3}}{2}$

2. $\tilde{a} \cdot \tilde{c} = 3 \times 4 \times \cos 90^\circ$
$= 0$

3. $\tilde{a} \cdot \tilde{d} = 3 \times 2 \times \cos 180^\circ$
$= -6$

4. $\tilde{b} . \tilde{c} = 5 \times 4 \times \cos 60^\circ$
$= 10$

5. $\tilde{b} \cdot \tilde{d} = 5 \times 2 \times \cos 150^\circ$
$= -5\sqrt{3}$

6. $\tilde{c} \cdot \tilde{d} = 4 \times 2 \times \cos 90^\circ$
$= 0$

7. $\tilde{e} \cdot \tilde{f} = 2 \times 3 \times \cos 90^\circ$
$= 0$

8. $\tilde{f} \cdot \tilde{e} = 3 \times 2 \times \cos 90^\circ$
$= 0^\circ$

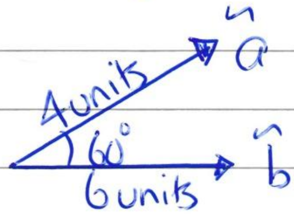
9. $\tilde{e} \cdot \tilde{e} = 2 \times 2 \times \cos 0^\circ$
$= 4$

10. $\tilde{f} \cdot \tilde{f} = 3 \times 3 \times \cos 0^\circ$
$= 9$

11. $\tilde{f} \cdot \tilde{g} = 3 \times 4 \times \cos 45^\circ$
$= 6\sqrt{2}$

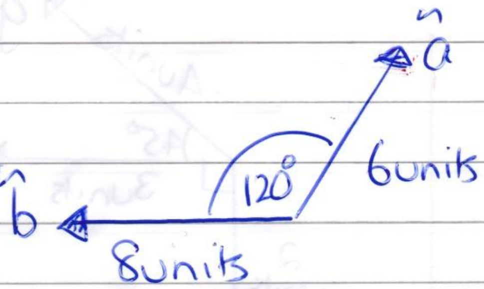
12. $\tilde{g} \cdot \tilde{f} = 4 \times 3 \times \cos 45^\circ$
$= 6\sqrt{2}$

13.



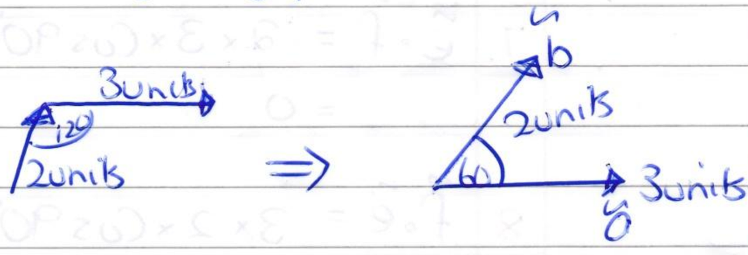
$\tilde{a} \cdot \tilde{b} = 4 \times 6 \times \cos 60$
$= 12$

14.



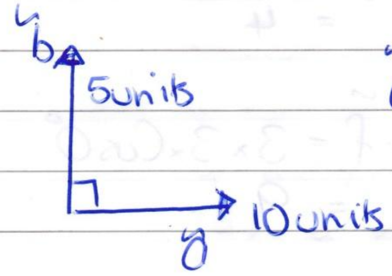
$\hat{a} \cdot \hat{b} = 6 \times 8 \times \cos 120°$
$= -24$

15.



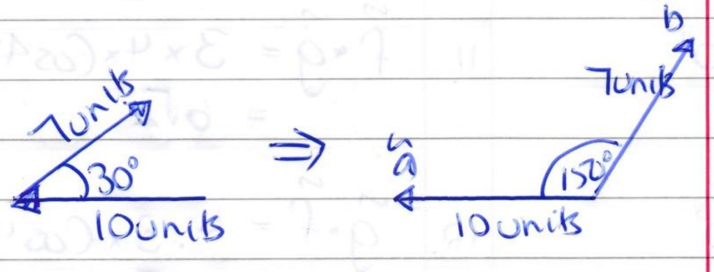
$\hat{a} \cdot \hat{b} = 3 \times 2 \times \cos 60$
$= 3$

16



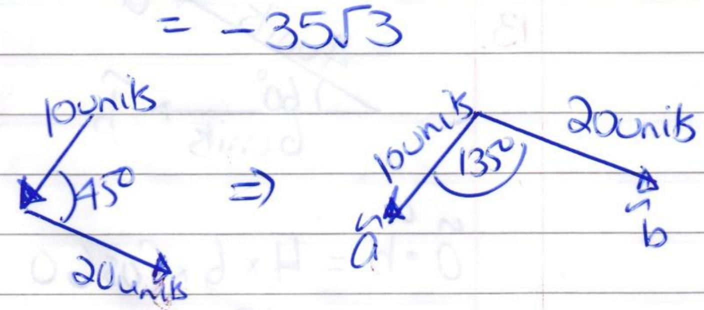
$\hat{a} \cdot \hat{b} = 10 \times 5 \times \cos 90$
$= 0$

17.



$\hat{a} \cdot \hat{b} = 10 \times 7 \times \cos 150°$
$= -35\sqrt{3}$

18.



$\hat{a} \cdot \hat{b} = 10 \times 20 \times \cos 135°$
$= -100\sqrt{2}$

19 a) $|\hat{a}|$
magnitude of a
= Scalar

b) $\hat{a} \cdot \hat{b}$ ⇒ dot product
⇒ scalar

c) $\hat{a} + \hat{b}$ ⇒ add 2 vectors
⇒ vector

d) $\hat{a} - \hat{b}$ ⇒ subtract 2 vectors
⇒ vector

e) $\hat{a} + 2\hat{b}$ ⇒ vector

f) $\hat{a} \cdot (2\hat{b})$
$= 2\hat{a} \cdot \hat{b}$
= scalar

g) $(\hat{a} + \hat{b}) \cdot (\hat{c} + \hat{d})$
$\downarrow \qquad \downarrow$
$\hat{e} \cdot \hat{f}$
⇒ scalar

h) $|\hat{a} + \hat{b}|$ = magnitude
= scalar

i) $\hat{a} + \lambda\hat{b}$ ⇒ vector

j) $\hat{a} \cdot \lambda\hat{b} = \lambda\hat{a} \cdot \hat{b}$
= scalar

20. a) $\hat{i} \cdot \hat{i}$ $\quad \xrightarrow{\hat{i}} \quad \xrightarrow{\hat{i}}$

$\Rightarrow 1 \times 1 \times \cos 0^\circ$
$= 1$

b) $\hat{i} \cdot \hat{j}$ $\quad \rightarrow \hat{i} \quad \uparrow \hat{j}$

$\Rightarrow 1 \times 1 \times \cos 90^\circ$
$= 0$

c) $\hat{j} \cdot \hat{j}$ $\quad \uparrow \hat{j} \quad \uparrow \hat{j}$

$\Rightarrow 1 \times 1 \times \cos 0^\circ$
$= 1$

21. a) $(\hat{a}+\hat{b}) \cdot (\hat{a}-\hat{b})$

$= \hat{a}\cdot\hat{a} - \cancel{\hat{a}\cdot\hat{b}} + \cancel{\hat{b}\cdot\hat{a}} - \hat{b}\cdot\hat{b}$

$= a^2 - b^2$

b) $(\hat{a}+\hat{b}) \cdot (\hat{a}+\hat{b})$
$= \hat{a}\cdot\hat{a} + \hat{a}\cdot\hat{b} + \hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{b}$

$= a^2 + 2\hat{a}\cdot\hat{b} + b^2$

c) $(\hat{a}-\hat{b}) \cdot (\hat{a}-\hat{b})$

$= \hat{a}\cdot\hat{a} - \hat{a}\cdot\hat{b} - \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b}$

$= a^2 - 2\hat{a}\cdot\hat{b} + b^2$

d) $(2\hat{a}+\hat{b}) \cdot (2\hat{a}-\hat{b})$
$= 4\hat{a}\cdot\hat{a} - \cancel{2\hat{a}\cdot\hat{b}} + \cancel{2\hat{a}\cdot\hat{b}}$
$- \hat{b}\cdot\hat{b}$

$= 4a^2 - b^2$

e) $(\hat{a}+3\hat{b}) \cdot (\hat{a}-2\hat{b})$
$= \hat{a}\cdot\hat{a} - \hat{a}\cdot 2\hat{b} + 3\hat{b}\cdot\hat{a} - 6\hat{b}\cdot\hat{b}$

$= a^2 + 2\hat{a}\cdot\hat{b} - 6b^2$

f) $\hat{a}\cdot(\hat{a}-\hat{b}) + \hat{a}\cdot\hat{b}$

$= \hat{a}\cdot\hat{a} - \cancel{\hat{a}\cdot\hat{b}} \cancel{+ \hat{a}\cdot\hat{b}}$

$= a^2$

22. $\hat{a}$ & $\hat{b}$ are $\perp$
ie $\hat{a}\cdot\hat{b} = 0$

$(\hat{a}+\hat{b}) \cdot (\hat{a}-2\hat{b})$
$= \hat{a}\cdot\hat{a} - \hat{a}\cdot 2\hat{b} + \hat{b}\cdot\hat{a}$
$- 2\hat{b}\cdot\hat{b}$

$= a^2 - \underbrace{\cancel{2\hat{a}\cdot\hat{b}}}_{0} + \underbrace{\cancel{\hat{a}\cdot\hat{b}}}_{0} - 2b^2$

$= a^2 - 2b^2$

23. $\hat{a}$ & $\hat{b}$ are ⊥ ie $\hat{a}\cdot\hat{b}=0$

a) $\hat{a}=\hat{b}$ not true
if $\hat{a}=\hat{b}$ then they are parallel vectors

b) $\hat{a}\cdot\hat{b}=0$ yes true as $\hat{a}$ & $\hat{b}$ are ⊥ ∴ angle between them is $90°$ & $\cos 90=0$

c) $ab=0$ not true as this implies magnitudes & would suggest that either $\hat{a}$ or $\hat{b}$ has a magnitude of 0 which doesn't imply that they are ⊥

d) $\hat{a}\cdot(\hat{a}+\hat{b})=a^2$

$\hat{a}\cdot\hat{a}+\hat{a}\cdot\hat{b}$

$a^2+0 \Rightarrow a^2$ yes true

24. $\hat{a}$ is ⊥ to $(\hat{b}-\hat{c})$

a) $\hat{a}\cdot(\hat{b}-\hat{c})=0$ yes true

as ⊥ implies $\cos 90=0$

b) $\hat{a}\cdot\hat{b}=\hat{a}\cdot\hat{c}$

we know
$\hat{a}\cdot(\hat{b}-\hat{c})=0$
$\Rightarrow \hat{a}\cdot\hat{b}-\hat{a}\cdot\hat{c}=0$

ie $\hat{a}\cdot\hat{b}=\hat{a}\cdot\hat{c}$
true

c) $\hat{b}=\hat{c}$
if we know $\hat{b}-\hat{c}$

this means
$\hat{b}-\hat{b}$ or $\hat{c}-\hat{c}$
which implies 0 vector & a zero vector doesn't imply ⊥

d) $\hat{a}$ is ⊥ to $\hat{c}-\hat{b}$

we know $\hat{a}\cdot(\hat{b}-\hat{c})=0$
so $\hat{a}\cdot\hat{b}-\hat{a}\cdot\hat{c}=0$

$\hat{a}\cdot\hat{b}=\hat{a}\cdot\hat{c}$

$0=\hat{a}\cdot\hat{c}-\hat{a}\cdot\hat{b}$

$0=\hat{a}(\hat{c}-\hat{b})$

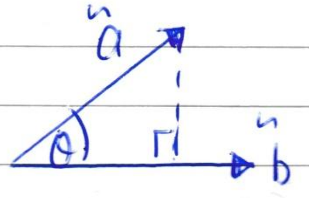
yes true.

25. $\hat{a} = x_1 \dot{i} + y_1 \dot{j}$

$\hat{b} = x_2 \dot{i} + y_2 \dot{j}$

$\hat{a} \cdot \hat{b} = (x_1)(x_2) + (y_1)(y_2)$

26.



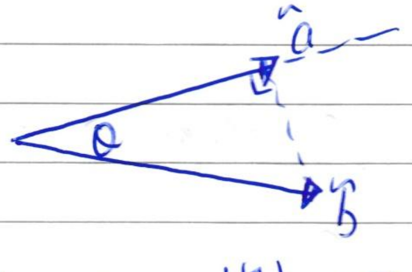
$\hat{a} \cdot \hat{b} = 14 \qquad |\hat{b}| = 5$

projection of $\hat{a}$ onto $\hat{b}$

$|\hat{a}| \cos\theta \, \hat{b} \qquad \hat{b}$ = unit vector

$14 \times \frac{1}{5} \Rightarrow 2{\cdot}8$

27.



$\hat{a} \cdot \hat{b} = 18 \qquad |\hat{a}| = 25$

projection of $\hat{b}$ onto $\hat{a}$

$|\hat{b}| \cos\theta \, \hat{a} \qquad \hat{a}$ = unit vector

$18 \times \frac{1}{25} = 0{\cdot}72$

28. if $\hat{a}$ & $\hat{b}$ are $\perp$

ie $\hat{a} \cdot \hat{b} = 0$

a) $\hat{a} \cdot (\hat{a} - \hat{b}) = 0$

$\hat{a} \cdot \hat{a} - \hat{a} \cdot \hat{b}$

$\hat{a} - 0 \neq 0$

No!

b) $(\hat{a} + \hat{b}) \cdot (\hat{a} - \hat{b}) = 0$

$\hat{a} \cdot \hat{a} \cancel{- \hat{a} \cdot \hat{b}} \cancel{+ \hat{a} \cdot \hat{b}} - \hat{b} \cdot \hat{b}$

$\hat{a}^2 - \hat{b}^2 \neq 0$

unless $|a| = |b|$

No!

c) $(\hat{a} + \hat{b}) \cdot (\hat{a} + \hat{b}) = \hat{a}^2 + \hat{b}^2$

$\hat{a} \cdot \hat{a} + \underbrace{\cancel{\hat{a} \cdot \hat{b}}}_{0} + \underbrace{\cancel{\hat{a} \cdot \hat{b}}}_{0} + \hat{b} \cdot \hat{b}$

$\hat{a}^2 + \hat{b}^2 = \hat{a}^2 + \hat{b}^2$ yes

29. $|\hat{a}| = 5 \quad |\hat{b}| = 3 \quad \hat{a} \cdot \hat{b} = 7$

a) $\hat{a} \cdot \hat{b} = |\hat{a}| \times |\hat{b}| \times \cos\theta$

$7 = 5 \times 3 \times \cos\theta$

$\cos\theta = \frac{7}{15}$

$\theta = 62^\circ$

29b) $\hat{a} \cdot \hat{a}$
$= a^2$
$= 5^2$
$= 25$

c) $\hat{b} \cdot \hat{b}$
$= b^2$
$= 3^2$
$= 9$

d) $(\hat{a}-\hat{b}) \cdot (\hat{a}-\hat{b})$

$= \hat{a}\cdot\hat{a} - \hat{a}\cdot\hat{b} - \hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{b}$

$= 25 - 7 - 7 + 9$
$= 20$

e) $|\hat{a}-\hat{b}| \Rightarrow (\hat{a}-\hat{b}) \cdot (\hat{a}-\hat{b})$
$\Rightarrow \sqrt{25-7-7+9}$
$= \sqrt{20}$
$= 2\sqrt{5}$

30. $\hat{p} = 3\hat{a} + 2\hat{b} \quad \hat{q} = 4\hat{a} - \hat{b}$

$|\hat{a}| = 3 \quad |\hat{b}| = 2 \quad \hat{a}\cdot\hat{b} = 0$

$\hat{p}\cdot\hat{q} = (3\hat{a} + 2\hat{b}) \cdot (4\hat{a} - \hat{b})$

$= 12\hat{a}\cdot\hat{a} - \cancel{3\hat{a}\cdot\hat{b}} + \cancel{8\hat{a}\cdot\hat{b}} - 2\hat{b}\hat{b}$

$12\hat{a}^2 - 2\hat{b}^2$
$= 12(9) - 2(4)$

$= 100$

* note $\hat{a}^2$ is prior to the $\sqrt{\ }$ when you are finding the magnitude

31 $\hat{a}\cdot(\hat{b}+\hat{c})$
$= \hat{a}\cdot\hat{b} + \hat{a}\cdot\hat{c}$

scalar + scalar = scalar

$\hat{a}\cdot(\hat{b}-\hat{c})$
$= \hat{a}\cdot\hat{b} - \hat{a}\cdot\hat{c}$
scalar − scalar ⇒ scalar

$\hat{a}\cdot(\hat{b}\cdot\hat{c})$
scalar

$\hat{a}$ · scalar ⇒ vector.

$\hat{a}\cdot\hat{b}\cdot\hat{c}$
doesn't give a scalar

& dot product ⇒ scalar

∴ has no meaning to find $\hat{a}\cdot\hat{b}\cdot\hat{c}$

32 a) $|\hat{a}\cdot\hat{b}| \leq |\hat{a}| \times |\hat{b}|$

we know
$\hat{a}\cdot\hat{b} = |\hat{a}||\hat{b}|\cos\theta$

$-1 \leq \cos\theta \leq 1$

if we take the absolute value

ie $|\hat{a}\cdot\hat{b}| = |\hat{a}| \times |\hat{b}| \times |\cos\theta|$

exist only between 0 & 1

ie $|\hat{a}| \times |\hat{b}|$ will always be equal to or less than $|\hat{a}\cdot\hat{b}|$

ie $|\hat{a}\cdot\hat{b}| \leq |\hat{a}| \times |\hat{b}|$

b) $(\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b})$

$= \hat{a}\cdot\hat{a} + \hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b}$

$= a^2 + 2\hat{a}\cdot\hat{b} + b^2$

note $(\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b}) = |\hat{a}+\hat{b}|$

ie $|\hat{a}+\hat{b}| = a^2 + 2\hat{a}\cdot\hat{b} + b^2$ (1)

we know already

$\hat{a}\cdot\hat{b} \leq |\hat{a}| \times |\hat{b}|$

$\therefore$ $2\hat{a}\cdot\hat{b} \leq 2|\hat{a}| \times |\hat{b}|$

note

$(|\hat{a}| + |\hat{b}|)^2$

$= (|\hat{a}| + |\hat{b}|)(|\hat{a}| + |\hat{b}|)$

$= a^2 + 2|\hat{a}||\hat{b}| + b^2$ (2)

ie $\underset{(1)}{|\hat{a}+\hat{b}|} \leq \underset{(2)}{|\hat{a}| + |\hat{b}|}$

$a^2 + 2\hat{a}\cdot\hat{b} + b^2 \leq a^2 + 2|\hat{a}||\hat{b}| + b^2$

as all magnitudes are +ve & $\hat{a}\cdot\hat{b}$ could be -ve.

it implies that

$|\hat{a}+\hat{b}| \leq |\hat{a}| + |\hat{b}|$

Ex 8B.

1. $\underset{\sim}{a} = 3i - 2j \quad \underset{\sim}{b} = 5i + 6j \quad \underset{\sim}{c} = 2i - j$

a) $\underset{\sim}{a} \cdot \underset{\sim}{b} = (3)(5) + (-2)(6)$
$= 3$

b) $\underset{\sim}{b} \cdot \underset{\sim}{a} = (5)(3) + (6)(-2)$
$= 3$

c) $\underset{\sim}{a} \cdot \underset{\sim}{c} = (3)(2) + (-2)(-1)$
$= 8$

d) $\underset{\sim}{b} \cdot \underset{\sim}{c} = (5)(2) + (6)(-1)$
$= 4$

2. $\underset{\sim}{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \underset{\sim}{y} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \quad \underset{\sim}{z} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

a) $\underset{\sim}{x} \cdot \underset{\sim}{y} = (2)(5) + (3)(-1)$
$= 7$

b) $\underset{\sim}{x} \cdot \underset{\sim}{z} = (2)(4) + (3)(2)$
$= 14$

c) $\underset{\sim}{z} \cdot \underset{\sim}{x} = (4)(2) + (2)(3)$
$= 14$

d) $\underset{\sim}{y} \cdot \underset{\sim}{z} = (5)(4) + (-1)(2)$
$= 18$

3. $\underset{\sim}{p} = \langle 3, 1 \rangle \quad \underset{\sim}{q} = \langle 2, -1 \rangle$
$\underset{\sim}{r} = \langle 5, 2 \rangle$

a) $\underset{\sim}{q} \cdot \underset{\sim}{r} = (2)(5) + (-1)(2)$
$= 8$

b) $2\underset{\sim}{q} \cdot 3\underset{\sim}{r} = 6\underset{\sim}{q} \cdot \underset{\sim}{r}$
$= 6(8)$
$= 48$

c) $\underset{\sim}{p} \cdot (\underset{\sim}{q} + \underset{\sim}{r})$

$= \langle 3, 1 \rangle \cdot (\langle 7, 1 \rangle$
$= (3)(7) + (1)(1)$
$= 22$

d) $\underset{\sim}{p} \cdot (\underset{\sim}{q} - \underset{\sim}{r})$
$\langle 3, 1 \rangle \cdot \langle -3, -3 \rangle$
$= (3)(-3) + (1)(-3)$
$= -12$

4. a) $(2i + 3j) \cdot (4i - 2j)$
$(2)(4) + (3)(-2)$
$= 8 - 6$
$\neq 0$ not $\perp$

b) $(-2i + j) \cdot (4i - 2j)$
$(-2)(4) + (1)(-2)$
$-8 + -2$
$\neq 0$
not $\perp$

c) $(3i-j)\cdot(2i+6j)$

$(3)(2)+(-1)(6)$

$=0$ yes $\perp$

d) $\langle 12,-3\rangle\cdot\langle 1,4\rangle$

$(12)(1)+(-3)(4)$

$=0$ yes $\perp$

e) $\begin{pmatrix}5\\2\end{pmatrix}\cdot\begin{pmatrix}-3\\7\end{pmatrix}$

$=(5)(-3)+(2)(7)$

$\neq 0$ $\therefore$ not $\perp$

f) $\begin{pmatrix}14\\8\end{pmatrix}\cdot\begin{pmatrix}-4\\7\end{pmatrix}$

$=(14)(-4)+(8)(7)$

$=0$ yes $\perp$

5. $\hat{d}=3i+j$ $\quad\hat{e}=2i+4j$

$\hat{f}=-2i-3j$

a) $\hat{d}\cdot\hat{e}=(3)(2)+(1)(4)$

$=10$

b) $\hat{e}\cdot\hat{f}=(2)(-2)+(4)(-3)$

$=-16$

c) $\hat{d}\cdot(\hat{e}+\hat{f})$

$(3i+j)\cdot(0i+j)$

$=(3)(0)+(1)(1)$

$=1$

d) $(\hat{d}+\hat{e})\cdot\hat{f}$

$(5i+5j)\cdot(-2i-3j)$

$=(5)(-2)+(5)(-3)$

$=-25$

6. $\hat{a}=2i-j$ $\quad\hat{b}=3i+2j$

$\hat{c}=4i-3j$

a) $\hat{a}\cdot(\hat{b}+\hat{c})$

$(2i-j)\cdot(7i-j)$

$=(2)(7)+(-1)(-1)$

$=15$

b) $(\hat{a}+\hat{b})\cdot\hat{c}$

$(5i+j)\cdot(4i-3j)$

$=(5)(4)+(1)(-3)$

$=17$

c) $\hat{b}\cdot(\hat{a}+\hat{c})$

$(3i+2j)\cdot(6i-4j)$

$=(3)(6)+(2)(-4)$

$=10$

d) $(\hat{a}-\hat{b})\cdot(\hat{b}-\hat{c})$

$=(-i-3j)\cdot(-i+5j)$

$=(-1)(-1)+(-3)(5)$

$=-14.$

7. $\hat{a} = 2i + 3j \quad \hat{b} = i - 4j$

$\hat{c} = -4i + 5j$

a) $\hat{a} \cdot \hat{b} = (2)(1) + (3)(-4)$
$= -10$

b). $\hat{a} \cdot \hat{c} = (2)(-4) + (3)(5)$
$= 7$

c) $\hat{b} + \hat{c} = (1-4)i + (-4+5)j$
$= -3i + j$

d) $\hat{a} \cdot (\hat{b} + \hat{c})$
$(2 + 3j) \cdot (-3i + j)$
$= (2)(-3) + (3)(1)$
$= -3$

$\hat{a} \cdot \hat{b} = -10 \quad \hat{a} \cdot \hat{c} = 7$

$\hat{a} \cdot (\hat{b} + \hat{c}) = \hat{a} \cdot \hat{b} + \hat{a} \cdot \hat{c}$
$-3 = -10 + 7$
$-3 = -3$ yes true.

8. $\hat{p} = 3i + 4j \quad \hat{q} = 5i - 12j$

a) $|\hat{p}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

b) $|\hat{q}| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

c) $\hat{p} \cdot \hat{q} = (3)(5) + (4)(-12)$
$= -33$

d) $\cos\theta = \dfrac{\hat{p} \cdot \hat{q}}{|\hat{p}| \times |\hat{q}|}$

$\cos\theta = \dfrac{-33}{5 \times 13}$

$\theta = \cos^{-1}\left(\dfrac{-33}{65}\right)$

$\approx 121°$

9. $\hat{c} = 7i + 7j \quad \hat{d} = 15i - 8j$

a) $|\hat{c}| = \sqrt{7^2 + 7^2} = 7\sqrt{2}$

b) $|\hat{d}| = \sqrt{15^2 + 8^2} = 17$

c) $\hat{c} \cdot \hat{d} = (7)(15) + (7)(-8)$
$= 49$

d) $\cos\theta = \dfrac{49}{7\sqrt{2} \times 17}$

$\theta = \cos^{-1}\left(\dfrac{49}{119\sqrt{2}}\right)$

$\theta \approx 73°$

10. $\hat{a} = (2i - 3j) \quad \hat{b} = 4i + 5j$
$\hat{c} = 2i - j$

$\hat{a} \cdot (\hat{b} + \hat{c}) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

$= (2)(6) + (-3)(4) = 0$

i.e $\perp$

11. $\tilde{a} = -2i + 2j \quad \tilde{b} = 5i + 2j$

$\tilde{c} = 4i - j$

$\tilde{a} + 2\tilde{b} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 10 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$

$\tilde{b} - 2\tilde{c} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$(\tilde{a} + 2\tilde{b}) \cdot (\tilde{b} - 2\tilde{c})$

$= (8)(-3) + (6)(4)$

$= 0 \quad \therefore \perp$

12. $\tilde{a} = 3i + 4j \quad \tilde{b} = 4i + 3j$

$\tilde{a} \cdot \tilde{b} = (3)(4) + (4)(3) = 24$

$|\tilde{a}| = \sqrt{3^2 + 4^2} = 5$

$|\tilde{b}| = \sqrt{4^2 + 3^2} = 5$

$\cos\theta = \frac{24}{5 \times 5}$

$\theta = \cos^{-1}\left(\frac{24}{25}\right)$

$\theta \approx 16^\circ$

13. $\tilde{c} = (24i + 7j) \quad \tilde{d} = (5i + 12j)$

$\tilde{c} \cdot \tilde{d} = (24)(5) + (7)(12) = 204$

$|\tilde{c}| = \sqrt{24^2 + 7^2} = 25 \quad |\tilde{d}| = \sqrt{5^2 + 12^2} = 13$

$\theta = \cos^{-1}\left(\frac{204}{25 \times 13}\right) \quad \theta \approx 51^\circ$

14. $\tilde{e} = 2i + j \quad \tilde{f} = 3i - 2j$

$\tilde{e} \cdot \tilde{f} = (2)(3) + (1)(-2) = 4$

$|\tilde{e}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$|\tilde{f}| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\theta = \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{13}}\right)$

$\theta \approx 60^\circ$

15. $\tilde{g} = 2i + j \quad \tilde{h} = -4i + 8j$

$\tilde{g} \cdot \tilde{h} = (2)(-4) + (1)(8)$

$= 0 \quad * \perp$

$|\tilde{g}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$|\tilde{h}| = \sqrt{4^2 + 8^2} = \sqrt{80}$

$\theta = \cos^{-1}\left(\frac{0}{\sqrt{5}\sqrt{80}}\right)$

$\theta = 90^\circ$

16. $\tilde{m} = -3i + 4j \quad \tilde{n} = 9i - 12j$

$\tilde{m} \cdot \tilde{n} = (-3)(9) + (4)(-12)$

$= -75$

$|\tilde{m}| = \sqrt{3^2 + 4^2} = 5$

$|\tilde{n}| = \sqrt{9^2 + 12^2} = 15$

$\theta = \cos^{-1}\left(\frac{-75}{5 \times 15}\right)$

$= 180^\circ$

17. $\tilde{p} = i + 0j \quad \tilde{q} = 12i - 5j$

$\tilde{p} \cdot \tilde{q} = (1)(12) + (0)(-5)$
$= 12$

$|\tilde{p}| = \sqrt{1^2 + 0^2} = 1$
$|\tilde{q}| = \sqrt{12^2 + 5^2} = 13$

$\theta = \cos^{-1}\left(\frac{12}{1 \times 13}\right)$

$\theta = 23^{\circ}$

18 $\tilde{a} = 2i + 3j \quad \tilde{b} = \lambda i + 12j$
$\tilde{c} = \mu i - 7j$

$\tilde{a}$ is parallel to $\tilde{b}$

ie $\begin{pmatrix} 2 \\ 3 \end{pmatrix} = k \begin{pmatrix} \lambda \\ 12 \end{pmatrix}$

$3 = 12k \quad k = 1/4$
$\therefore 2 = 1/4 \lambda$
$8 = \lambda$

$\tilde{a}$ is $\perp$ to $\tilde{c}$

ie $\tilde{a} \cdot \tilde{c} = 0$
$(2)(\mu) + (3)(-7) = 0$
$2\mu - 21 = 0$
$2\mu = 21$
$\mu = 10.5$

19. $\tilde{d} = wi + j$
$\tilde{e} = -i + 7j$
$\tilde{f} = xi + 5j$

$\tilde{d}$ is $\perp$ to $\tilde{e}$
ie $\tilde{d} \cdot \tilde{e} = 0$
$(w)(-1) + (1)(7) = 0$
$-w + 7 = 0$
$w = 7$

$|\tilde{d}| = |\tilde{f}|$ & x is $-$ve

$w^2 + 1^2 = x^2 + 5^2$

Note $w = 7$.
ie $7^2 + 1^2 = x^2 + 5^2$
$50 - 25 = x^2$
$25 = x^2$
$\therefore x = \pm 5$
but x is $-$ve
$x = -5$

20.
a) project $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ onto $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= \frac{\tilde{a} \cdot \tilde{b}}{|\tilde{b}|}$

$= \frac{(3)(2) + (4)(1)}{\sqrt{2^2 + 1^2}}$

$= \frac{10}{\sqrt{5}}$

$= 2\sqrt{5}$

b) $3i+4j$ vector project to $2i+j$

$$\frac{\hat{a}\cdot\hat{b}}{\hat{b}\cdot\hat{b}} \times \hat{b}$$

$$= \frac{(3)(2)+(4)(1)}{(2)(2)+(1)(1)} \times (2i+j)$$

$$= \frac{10}{5}(2i+j)$$

$$= 4i+2j$$

c) Scalar project $2i+j$ to $3i+4j$

$$\frac{\hat{a}\cdot\hat{b}}{|\hat{b}|} = \frac{(2)(3)+(1)(4)}{\sqrt{3^2+4^2}}$$

$$= \frac{10}{5} = 2$$

d) vector project $2i+j$ to $3i+4j$

$$\frac{\hat{a}\cdot\hat{b}}{\hat{b}\cdot\hat{b}} \times \hat{b}$$

$$\frac{(2)(3)+(1)(4)}{(3)(3)+(4)(4)} \times (3i+4j)$$

$$\frac{10}{25}(3i+4j)$$

$$= \frac{6}{5}i + \frac{8}{5}j$$

$$= 1{\cdot}2i + 1{\cdot}6j$$

21. $\perp$ to $3i-4j$ & of size 25

$$\sqrt{3^2+4^2} = 5$$

ie need 5 times as long

$$\binom{a}{b}\cdot\binom{3}{-4} = 0$$

ie $4i+3j$ or $-4i-3j$

ie $\pm 5(4i+3j)$

$\pm(20i+15j)$

22. $\perp$ to $2i+j$ but unit vector!

$$\sqrt{2^2+1^2} = \sqrt{5}$$

need $\frac{1}{\sqrt{5}}$ of length

$$\binom{a}{b}\cdot\binom{2}{1} = 0$$

ie $-i+2j$ or $i-2j$

ie $\frac{\pm 1}{\sqrt{5}}(i-2j)$

23. $\vec{OA} = 2i + 4j \quad \vec{OB} = 6i + 6j$

$\vec{OC} = 7i + 2j \quad \vec{OD} = 4i + j$

$\vec{AC} = \vec{AO} + \vec{OC}$

$= \begin{pmatrix}-2\\-4\end{pmatrix} + \begin{pmatrix}7\\2\end{pmatrix} = \begin{pmatrix}5\\-2\end{pmatrix}$

$\vec{BD} = \vec{BO} + \vec{OD}$

$= \begin{pmatrix}-6\\-6\end{pmatrix} + \begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}-2\\-5\end{pmatrix}$

$\vec{AC} \cdot \vec{BD} = \begin{pmatrix}5\\-2\end{pmatrix} \cdot \begin{pmatrix}-2\\-5\end{pmatrix}$

$= (5)(-2) + (-2)(-5)$

$= -10 + 10$

$= 0$

24. $\vec{OA} = \begin{pmatrix}4\\7\end{pmatrix} \quad \vec{OB} = \begin{pmatrix}6\\2\end{pmatrix} \quad \vec{OC} = \begin{pmatrix}8\\9\end{pmatrix}$

a) $\vec{AC} = \vec{AO} + \vec{OC} = \begin{pmatrix}-4\\-7\end{pmatrix} + \begin{pmatrix}8\\9\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$

b) $\vec{AB} = \vec{AO} + \vec{OB} = \begin{pmatrix}-4\\-7\end{pmatrix} + \begin{pmatrix}6\\2\end{pmatrix} = \begin{pmatrix}2\\-5\end{pmatrix}$

c) $\vec{AC} \cdot \vec{AB} = (4)(2) + (2)(-5) = -2$

d) $|\vec{AC}| = \sqrt{4^2 + 2^2} = \sqrt{20}$

$|\vec{AB}| = \sqrt{2^2 + 5^2} = \sqrt{29}$

$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{20}\sqrt{29}}\right)$

$\theta = 95^\circ$

25. $\hat{a} = 3i - j$

$\hat{b} = 4i + yj$

Angle between $\hat{a}$ & $\hat{b} = 45$

$\hat{a} \cdot \hat{b} = |\hat{a}||\hat{b}| \cos\theta$

$(3)(4) + (-1)(y) = 12 - y$

$|\hat{a}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$|\hat{b}| = \sqrt{4^2 + y^2}$

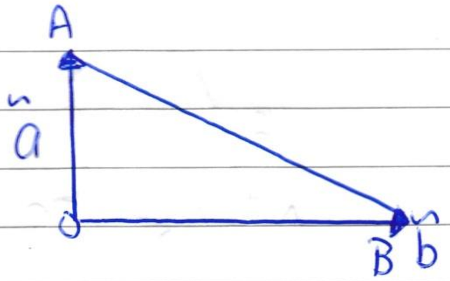
$12 - y = \sqrt{10}\sqrt{4^2 + y^2} \cos 45$

solve on calc

$y = -8$ or $y = 2$

Ex 8C.

1.



a) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\tilde{a} + \tilde{b}$

b) $\overrightarrow{AB} \cdot \overrightarrow{AB} = (AB)^2$

$(-\tilde{a} + \tilde{b}) \cdot (-\tilde{a} + \tilde{b})$

$= +a^2 - \cancel{\tilde{a}\tilde{b}} + \cancel{\tilde{a}\tilde{b}} + b^2$

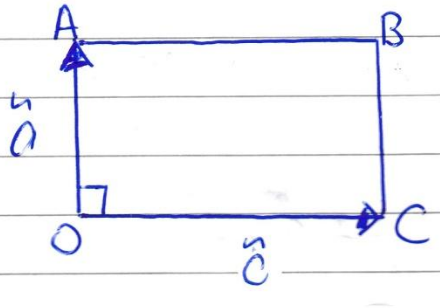
$= a^2 + b^2$

$(OA)^2 = \tilde{a} \cdot \tilde{a} = a^2$

$(OB)^2 = \tilde{b} \cdot \tilde{b} = b^2$

ie $a^2 + b^2$

2.



a) given OABC is a rectangle

$\tilde{a} \cdot \tilde{c} = 0$ ie $\perp$

b) $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\tilde{a} + \tilde{c}$

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$ $\quad ^{*}\overrightarrow{CB} = \overrightarrow{OA}$

$= \tilde{c} + \tilde{a}$

c) $|\overrightarrow{AC}| = \overrightarrow{AC} \cdot \overrightarrow{AC}$

$= (-\tilde{a} + \tilde{c}) \cdot (-\tilde{a} + \tilde{c})$

$= +a^2 - \tilde{a} \cdot \tilde{c} - \tilde{a} \cdot \tilde{c} + c^2$

$= c^2 + a^2 - 2\tilde{a} \cdot \tilde{c}$

$|\overrightarrow{OB}| = \overrightarrow{OB} \cdot \overrightarrow{OB}$

$= (\tilde{c} + \tilde{a}) \cdot (\tilde{c} + \tilde{a})$

$= c^2 + \tilde{a}\tilde{c} + \tilde{a} \cdot \tilde{c} + \tilde{a}$

$= c^2 + a^2 + 2\tilde{a} \cdot \tilde{c}$

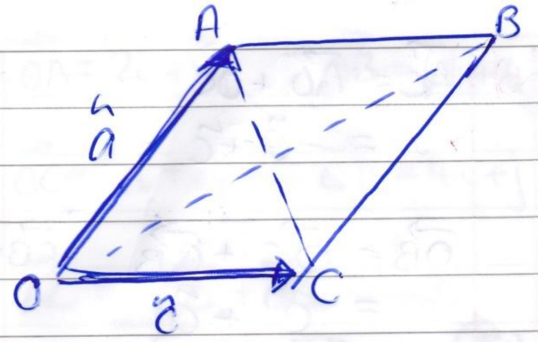
but $\tilde{a} \cdot \tilde{c} = 0$ as $\perp$

$\therefore |\overrightarrow{AC}| = c^2 + a^2$

$|\overrightarrow{OB}| = c^2 + a^2$

ie $|\overrightarrow{AC}| = |\overrightarrow{OB}|$

3.



OABC <u>is</u> a parallelogram

let $\vec{OA} = \tilde{a}$ $\quad \vec{OC} = \tilde{c}$

$\vec{AC} = \vec{AO} + \vec{OC}$
$= -\tilde{a} + \tilde{c}$

$\vec{OB} = \vec{OC} + \vec{CB}$ $\quad * \vec{CB} = \vec{OA}$

$= \tilde{c} + \tilde{a}$

If $\vec{AC} \perp \vec{OB}$

ie $(-\tilde{a}+\tilde{c}) \cdot (\tilde{c}+\tilde{a})$
$= -\cancel{\tilde{a}\cdot\tilde{c}} - a^2 + c^2 + \cancel{\tilde{a}\cdot\tilde{c}}$

$= c^2 - a^2$

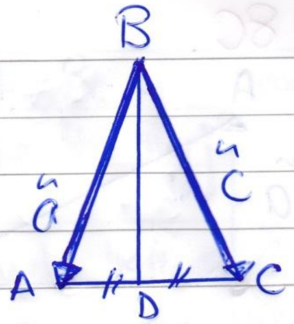
ie $|c| = |a|$ then

$c^2 - a^2 = 0$

ie for OABC to have diagonals cut at 90°, then $|a| = |c|$

ie OABC is a rhombus.

4.



ABC <u>is</u> isosceles where $AB = CB$
D <u>is</u> a midpoint of AC

a) $\vec{AC} = \vec{AB} + \vec{BC}$
$= -\tilde{a} + \tilde{c}$

$\vec{BD} = \vec{BA} + \vec{AD}$
$= \vec{BA} + \frac{1}{2}\vec{AC}$
$= \tilde{a} + \frac{1}{2}(-\tilde{a} + \tilde{c})$
$= \frac{1}{2}\tilde{a} + \frac{1}{2}\tilde{c}$

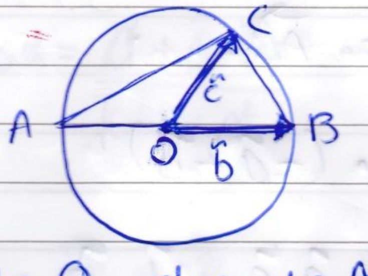
b) $\vec{AC} \cdot \vec{BD}$
$= (-\tilde{a} + \tilde{c}) \cdot (\frac{1}{2}\tilde{a} + \frac{1}{2}\tilde{c})$
$= -\frac{1}{2}a^2 - \cancel{\frac{1}{2}\tilde{a}\cdot\tilde{c}} + \cancel{\frac{1}{2}\tilde{a}\cdot\tilde{c}} + \frac{1}{2}c^2$

$= -\frac{1}{2}a^2 + \frac{1}{2}c^2$

we know ABC is isosceles $\therefore |a| = |c|$

ie $-\frac{1}{2}a^2 + \frac{1}{2}c^2$
$= 0$

ie $\perp$ $\therefore \triangle BDA$ is right angled.

5. 

centre O diameter AB

a) $\vec{CB} = \vec{CO} + \vec{OB}$
$= -\tilde{c} + \tilde{b}$

$\vec{AO} = \tilde{b}$ $\quad * \ \vec{AO} = \vec{OB}$

$\vec{AC} = \vec{AO} + \vec{OC}$
$= \tilde{b} + \tilde{c}$

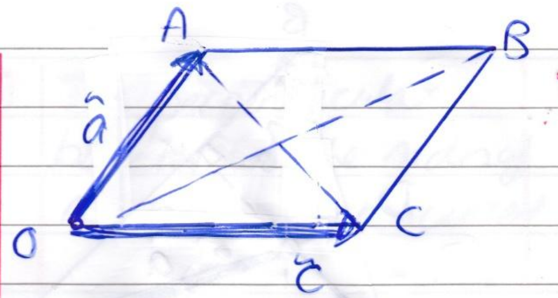
b) $\vec{AC} \cdot \vec{CB}$
$= (\tilde{b} + \tilde{c}) \cdot (-\tilde{c} + \tilde{b})$
$= \cancel{-\tilde{b} \cdot \tilde{c}} + b^2 - c^2 + \cancel{\tilde{b} \cdot \tilde{c}}$
$= b^2 - c^2$

but $|b| = |c|$ radii of Same circle

ie $b^2 - c^2$
$= 0$

ie $\vec{AC} \perp \vec{CB}$

∴ angle in a semi circle is right angled.

6. 

let OABC be a parallelogram

let $\vec{OA} = \tilde{a}$ $\quad \vec{OC} = \tilde{c}$

need to prove
$(OB)^2 + (AC)^2 = (OA)^2 + (AB)^2 + (BC)^2 + (OC)^2$

$\vec{OB} = \vec{OA} + \vec{AB} = \tilde{a} + \tilde{c}$
$\vec{AC} = \vec{AO} + \vec{OC} = -\tilde{a} + \tilde{c}$

$(OB)^2 \Rightarrow \vec{OB} \cdot \vec{OB} = (\tilde{a} + \tilde{c}) \cdot (\tilde{a} + \tilde{c})$
$= a^2 + 2\tilde{a} \cdot \tilde{c} + c^2$

$(AC)^2 = \vec{AC} \cdot \vec{AC} = (-\tilde{a} + \tilde{c}) \cdot (-\tilde{a} + \tilde{c})$
$= a^2 - 2\tilde{a} \cdot \tilde{c} + c^2$

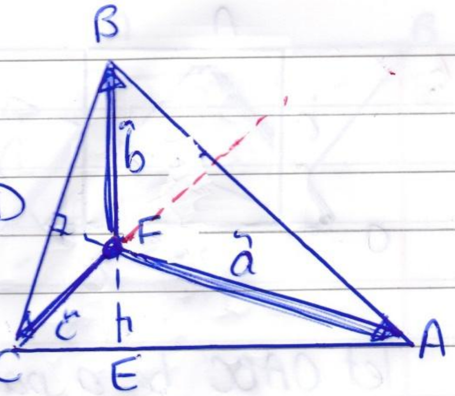
$(OB)^2 + (AC)^2$
$= a^2 + 2\tilde{a} \cdot \tilde{c} + c^2 + a^2 - 2\tilde{a} \cdot \tilde{c} + c^2$

$= 2a^2 + 2c^2$

$(OA)^2 + (AB)^2 + (BC)^2 + (CO)^2$
$= a^2 + c^2 + a^2 + c^2$
$= 2a^2 + 2c^2$

LHS = RHS as required.

7.



We are given the $\perp$

BE $\perp$ AC & AD $\perp$ BC

given $\hat{a} \cdot \vec{BC} = 0 \quad -(1)$
$\hat{b} \cdot \vec{AC} = 0 \quad -(2)$

$\vec{CF} = \hat{c}$

need to show $\vec{CF} \cdot \vec{AB} = 0$

$\vec{AB} = \vec{AF} + \vec{FB}$
$= -\hat{a} + \hat{b}$

$\vec{BC} = \vec{BF} + \vec{FC} = -\hat{b} + \hat{c}$

$\vec{AC} = \vec{AF} + \vec{FC} = -\hat{a} + \hat{c}$

from (1) $\hat{a} \cdot (-\hat{b} + \hat{c}) = 0$
$-\hat{a} \cdot \hat{b} + \hat{a} \cdot \hat{c} = 0$

from (2) $\hat{b} \cdot (-\hat{a} + \hat{c}) = 0$
$-\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} = 0$

ie $\hat{a} \cdot \hat{c} = \hat{a} \cdot \hat{b}$ & $\hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{c}$ (3)

$\vec{CF} \cdot \vec{AB}$

$\hat{c} \cdot (-\hat{a} + \hat{b})$

$\Rightarrow -\hat{a} \cdot \hat{c} + \hat{b} \cdot \hat{c}$

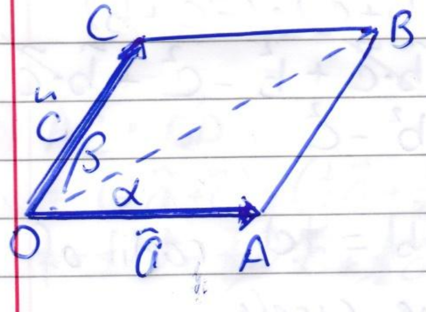
but from (3)

$\hat{a} \cdot \hat{c} = \hat{a} \cdot \hat{b}$

& $\hat{b} \cdot \hat{c} = \hat{a} \cdot \hat{b}$

$\therefore -\hat{a} \cdot \hat{b} + \hat{a} \cdot \hat{b}$
$= 0$
ie $\perp$

8.



OABC <u>is</u> a rhombus

a) $\vec{OB} = \vec{OA} + \vec{AB}$
$= \hat{a} + \hat{c}$

$$\cos\alpha = \frac{\vec{OB} \cdot \vec{OA}}{|\vec{OB}| \times |\vec{OA}|}$$

$$\cos\alpha = \frac{(\hat{a} + \hat{c}) \cdot (\hat{a})}{|\vec{OB}| \times a^2}$$

$$\cos\alpha = \frac{\hat{a} + \hat{a}\cdot\hat{c}}{|\vec{OB}| \times a^2}$$

$$\vec{OB} = \vec{OC} + \vec{CB}$$
$$= \hat{c} + \hat{a}$$

$$\cos\beta = \frac{\vec{OC}\cdot\vec{OB}}{|\vec{OC}| \times |\vec{OB}|}$$

$$\cos\beta = \frac{\hat{c}\cdot(\hat{c}+\hat{a})}{c^2 \times |\vec{OB}|}$$

$$= \frac{c^2 + \hat{a}\cdot\hat{c}}{c^2 + |\vec{OB}|}$$

b). OABC is a rhombus

so $|\hat{c}| = |\hat{a}|$ ie $a^2 = c^2$

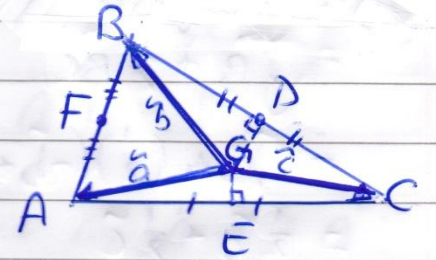
ie $\cos\alpha = \dfrac{a^2 + \hat{a}\cdot\hat{c}}{|\vec{OB}|\cdot a^2}$

& $\cos\beta = \dfrac{a^2 + \hat{a}\cdot\hat{c}}{|\vec{OB}|\cdot a^2}$

$\therefore \cos\alpha = \cos\beta$

ie $\alpha = \beta$.

9. The perpendicular bisectors of the sides of a triangle are concurrent



$AE = EC$, $BD = DC$, $AF = FB$

as shown.

given $|\hat{a}| = |\hat{b}| = |\hat{c}|$

a) $\vec{GE} = \vec{GA} + \vec{AE}$ $\quad * \ AE = \frac{1}{2}\vec{AC}$

$$\vec{AC} = \vec{AG} + \vec{GC}$$
$$= -\hat{a} + \hat{c}$$

$$\vec{GE} = \hat{a} + (1/2)(-\hat{a} + \hat{c})$$
$$= \tfrac{1}{2}\hat{a} + \tfrac{1}{2}\hat{c}$$

$\vec{AC}\cdot\vec{GE}$

$$\Rightarrow (-\hat{a} + \hat{c})\cdot(\tfrac{1}{2}\hat{a} + \tfrac{1}{2}\hat{c})$$
$$= -\tfrac{1}{2}a^2 - \cancel{\tfrac{1}{2}\hat{a}\cdot\hat{c}} + \cancel{\tfrac{1}{2}\hat{a}\cdot\hat{c}} + \tfrac{1}{2}c^2$$

$$= -\tfrac{1}{2}a^2 + \tfrac{1}{2}c^2$$

but we are given fact

so $-\frac{1}{2}a^2 + \frac{1}{2}c^2 = 0$

so $a^2 = c^2$

b. $\vec{BC} = \vec{BG} + \vec{GC}$
$= -\underset{\sim}{b} + \underset{\sim}{c}$

$\vec{GD} = \vec{GB} + \vec{BD}$
$= \vec{GB} + \frac{1}{2}\vec{BC}$
$= \underset{\sim}{b} + (\frac{1}{2})(-\underset{\sim}{b} + \underset{\sim}{c})$
$= \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}$

$\vec{BC} \cdot \vec{GD} = (-\underset{\sim}{b} + \underset{\sim}{c}) \cdot (\frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c})$

$\Rightarrow -\frac{1}{2}b^2 - \cancel{\frac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{c}} + \cancel{\frac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{c}} + \frac{1}{2}c^2$

$= -\frac{1}{2}b^2 + \frac{1}{2}c^2$

we know $\vec{BC} \cdot \vec{GD} = 0$

ie $-\frac{1}{2}b^2 + \frac{1}{2}c^2 = 0$
$\therefore b^2 = c^2$

c) $\vec{AB} = \vec{AG} + \vec{GB}$
$= -\underset{\sim}{a} + \underset{\sim}{b}$

$\vec{GF} = \vec{GA} + \vec{AF}$
$= \vec{GA} + \frac{1}{2}\vec{AB}$
$= \underset{\sim}{a} + (\frac{1}{2})(-\underset{\sim}{a} + \underset{\sim}{b})$
$= +\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

$\vec{AB} \cdot \vec{GF}$

$= (-\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b})$

$\Rightarrow -\frac{1}{2}a^2 + \cancel{\frac{1}{2}\underset{\sim}{a}\cdot\underset{\sim}{b}} - \cancel{\frac{1}{2}\underset{\sim}{a}\cdot\underset{\sim}{b}} + \frac{1}{2}b^2$

$= -\frac{1}{2}a^2 + \frac{1}{2}b^2$

but we know

$a^2 = c^2$ & $b^2 = c^2$

$\therefore -\frac{1}{2}c^2 + \frac{1}{2}c^2$
$= 0$

ie $\vec{AB} \cdot \vec{GF} = 0$

ie $AB \perp GF$.

Misc Chap 8.

1.a) ABCD is rhombus $\Longleftrightarrow$ ABC is parallelogram

False a rhombus $\rightarrow$ parallelogram

but parallelogram $\not\rightarrow$ rhombus

b) diagonals of PQRS cut at $90°$ $\Longleftrightarrow$ PQRS is a rhombus

False as kite diagonals also cut at $90°$, but it's not a rhombus

c) diagonals of parallelogram WXYZ cut at $90°$ $\Longleftrightarrow$ WXYZ is a rhombus

This is true as $\rightarrow$ is $\leftarrow$

2. if $\tilde{a}$ is $\perp$ to $(\tilde{b}-\tilde{a})$

ie $\tilde{a}\cdot(\tilde{b}-\tilde{a})=0$

ie $\tilde{a}\cdot\tilde{b}-\tilde{a}^2=0$

(a) $\tilde{a}\cdot(\tilde{b}-\tilde{a})=0$ true

(b) $\tilde{a}\cdot b-\tilde{a}\cdot\tilde{a}=0$

$\tilde{a}\cdot\tilde{b}=\tilde{a}\cdot\tilde{a}$

true.

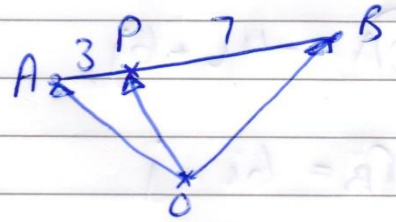
(c) $\tilde{a}=\tilde{b}$ not true.

as $\tilde{a}\cdot(\underbrace{\tilde{a}-\tilde{a}}_{\text{zero vector.}})$

(d) $\tilde{a}\cdot\tilde{b}=|a|^2$

true as $\tilde{a}\cdot\tilde{a}=a^2$.

3. $\vec{OA}=i+3j$ $\vec{OB}=21i-7j$



$\vec{AP}:\vec{PB}=3:7$

$\vec{OP}=\vec{OA}+\frac{3}{10}\vec{AB}$

$\vec{AB}=\vec{AO}+\vec{OB}$
$=(-i-3j)+(21i-7j)$
$=20i-10j$

$\vec{OP}=(i+3j)+\frac{3}{10}(20i-10j)$

$=i+3j+6i-3j$

$=7i+0j$

4. 8 letters
must have 2 vowels
6 consonants
no repeats.

$\binom{5}{2} \times \binom{21}{6}$

$= 542640$

* doesn't say to arrange the letters so only ^nC_r

5. $\vec{OA} = -4i + 6j$

${}_B r_A = 6i - j$

${}_C r_B = 4i + 5j$

a) $r_B - r_A = {}_B r_A$

$r_B = {}_B r_A + r_A$

$\vec{OB} = \begin{pmatrix} 6 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$_C r - r_B = {}_C r_B$

$_C r = {}_C r_B + r_B$

$\vec{OC} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$

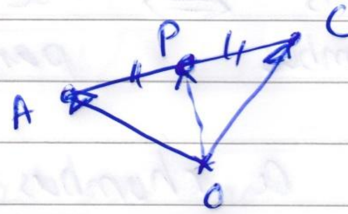
$\vec{AC} = \vec{AO} + \vec{OC}$

$= (4i - 6j) + (6i + 10j)$

$= 10i + 4j$

$|\vec{AC}| = \sqrt{100 + 16} = \sqrt{116} = 2\sqrt{29}$

b) $\vec{OC} = 6i + 10j$

c) 

$\vec{OP} = \vec{OA} + \frac{1}{2}\vec{AC}$

$= \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 10 \\ 4 \end{pmatrix}$

$= i + 8j$

6. 3 digit N° from 1, 2, 3, 4, 5

a) use multiple times

$\underline{5} \times \underline{5} \times \underline{5}$

$= 125$

b) once only each

$\underline{5} \times \underline{4} \times 3$

$= 60$

c) no repeats & even

$\underline{4} \times \underline{3} \times \underline{2}$

2 or 4

$= 24$

d) no repeats & odd

$\underline{4} \times \underline{3} \times \underline{3}$

1, 3, 5

$= 36$

e) no repeats, odd, >300

$\frac{3}{\uparrow \; 3,4,5} \times \frac{3}{} \times \frac{1}{\text{"1"}} = 9$

$\frac{2}{\uparrow \; 4,5} \times \frac{3}{} \times \frac{1}{\text{"3"}} = 6$

$\frac{2}{\uparrow \; 3,4} \times \frac{3}{} \times \frac{1}{\text{"5"}} = 6$

$\Rightarrow \underline{21}$

8. A B C D E F 6 files

a) $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

b) A is on extreme left

A $\times 5 \times 4 \times 3 \times 2 \times 1 = 120$

c) A B C D E F AB⇒BA

$5! \times 2 = 240$

d) A B C $3 \times 2 \times 1 = 6$

e) A B C $3 \times 2 \times 1$ shuffle ok.

$3! \times 3! = 36$

f) A B C D E F

no shuffle

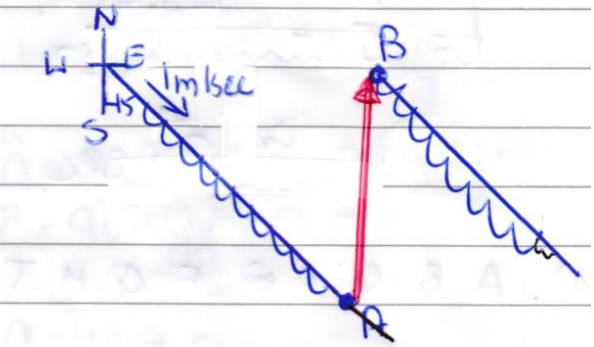
$3! = 6$

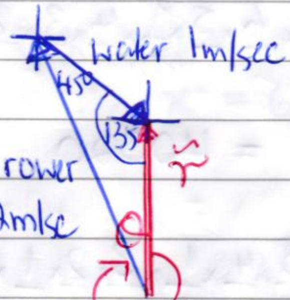
g) A B C D E F

Shuffle allowed

$4! \times 3! = 144$

9.





resultant.

$2^2 = 1^2 + r^2 - 2(1)(r)\cos(135^\circ)$

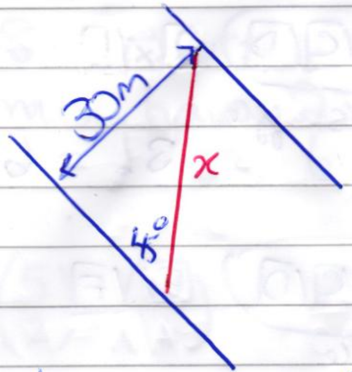
$r = 1.1637$ m/sec

$\frac{\sin\theta}{1} = \frac{\sin 135}{2}$

$\theta = 20.7^\circ$

$\therefore$ bearing $= 360 - 20.7^\circ$

$= 339^\circ$



distance: $x = \dfrac{30}{\sin 45}$

$x = 42.426\,m$

$t = \dfrac{d}{S} = \dfrac{42.426}{1.1637}$

≈ 36 sec

10. A B C D E F G H I

a) $\underline{8} \times \underline{7} \times \underline{6} \times \underline{5} \times \underset{"D"}{\underline{1}} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$

$= 8! = 40320$

b) $\underset{"E"}{\underline{1}} \times \underline{6} \times \underline{5} \times \underline{4} \times \underset{"D"}{\underline{1}} \times \underline{3} \times \underline{2} \times \underline{1} \times \underset{"H"}{\underline{1}}$

OR "H" ... "E"

$6! \times 2 = 1440$

{F C}
{C I}

c) $\underset{"E"}{\underline{1}} \times \underline{2} \times \underline{1} \; \underline{4} \times \underset{"D"}{\underline{1}} \times \underline{3} \times \underline{2} \times \underline{1} \times \underset{"H"}{\underline{1}}$

"H" ... "E"

CEI CEI CEI CEI — 4 options

$= 2 \times (2 \times 1 \times 4 \times 1 \times 3 \times 2 \times 1) \times 4$

$= 384$

11. $\vec{OA} = 2\hat{p} + \hat{q}$

$\vec{OB} = 3\hat{p} - \hat{q}$

$\vec{OC} = 6\hat{p} - 7\hat{q}$

Collinear

ie $\lambda \vec{AB} = \mu \vec{AC} = \gamma \vec{BC}$

$\vec{AB} = \begin{pmatrix} -2p \\ -q \end{pmatrix} + \begin{pmatrix} 3p \\ -q \end{pmatrix}$

$= \begin{pmatrix} p \\ -2q \end{pmatrix}$

$\vec{BC} = \begin{pmatrix} -3p \\ +q \end{pmatrix} + \begin{pmatrix} 6p \\ -7q \end{pmatrix}$

$= \begin{pmatrix} 3p \\ -6q \end{pmatrix} = 3\begin{pmatrix} p \\ -2q \end{pmatrix}$

$\vec{AC} = \begin{pmatrix} -2p \\ -q \end{pmatrix} + \begin{pmatrix} 6p \\ -7q \end{pmatrix}$

$= \begin{pmatrix} 4p \\ -8q \end{pmatrix} = 4\begin{pmatrix} p \\ -2q \end{pmatrix}$

ie A B C are collinear

$\vec{AB} : \vec{BC}$

$1 : 3$

$\vec{AB} : \vec{AC}$

$1 : 4$

12. 13 people 6 men 7 women
5 are choosen for photo
& then arranged

a) no restriction
$\binom{13}{5} \times 5! = 154440$

b) 2 men & 3 women
$\binom{6}{2}\binom{7}{3} \times 5!$
$= 63000$

13. $\vec{OD} = \begin{pmatrix}-3\\3\end{pmatrix}$ $\vec{OE} = \begin{pmatrix}3\\2\end{pmatrix}$ $\vec{OF} = \begin{pmatrix}8\\7\end{pmatrix}$

a) $\vec{ED} = \begin{pmatrix}-3\\-2\end{pmatrix} + \begin{pmatrix}-3\\3\end{pmatrix} = \begin{pmatrix}-6\\1\end{pmatrix}$

b) $\vec{EF} = \begin{pmatrix}-3\\-2\end{pmatrix} + \begin{pmatrix}8\\7\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix}$

c) $\vec{ED} \cdot \vec{EF} = \begin{pmatrix}-6\\1\end{pmatrix} \cdot \begin{pmatrix}5\\5\end{pmatrix}$
$= (-6)(5) + (1)(5)$
$= -25$

d) $\cos\theta = \dfrac{-25}{\sqrt{6^2+1^2} \times \sqrt{5^2+5^2}}$

$\cos\theta = \dfrac{-25}{\sqrt{37} \times \sqrt{50}}$ $\theta = 125.54$
$\approx 126°$

14. $\hat{a} = 5\sqrt{3}i + j$

$\hat{b} = 2\sqrt{3}i + \omega j$

$\theta = 60°$ ie $\cos 60 = \frac{1}{2}$

$\dfrac{\begin{pmatrix}5\sqrt{3}\\1\end{pmatrix} \cdot \begin{pmatrix}2\sqrt{3}\\\omega\end{pmatrix}}{\sqrt{(5\sqrt{3})^2 + 1^2} \times \sqrt{(2\sqrt{3})^2 + \omega^2}} = \dfrac{1}{2}$

solve on calc
$\omega = 8$ or $\omega = -\frac{14}{3}$

15. $\hat{a} = 2i - 3j$ $\hat{b} = xi + 4j$
$\hat{c} = 9i - yj$

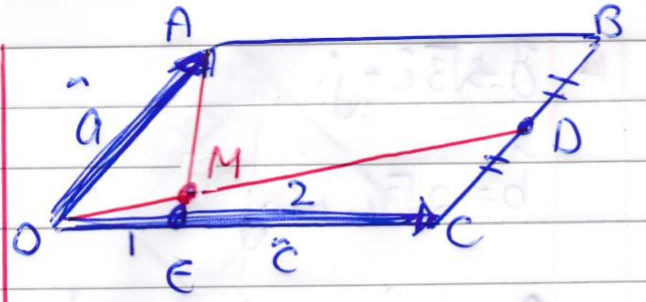
$\hat{a} \cdot \hat{b} = 0$
ie $(2)(x) + (-3)(4) = 0$
$\therefore x = 6$

$\hat{a} = \lambda\hat{c}$
ie $\begin{pmatrix}2\\-3\end{pmatrix} = \lambda\begin{pmatrix}9\\-y\end{pmatrix}$

$2 = \lambda 9$ $\lambda = 2/9$
$\therefore -3 = 2/9(-y)$
$\therefore y = 13.5$

16. $F = 6i + 4j$ $P = 2i - 7j$
$F + P = (8i - 3j)$ N
$|F| = \sqrt{36+16} = 2\sqrt{13} = 7.211$
$|P| = \sqrt{2^2+49} = \sqrt{53}$ 7.28
from calc
angle $\begin{bmatrix}8\\-3\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix} \Rightarrow 54.2°$

17.



$\vec{AM} = h\vec{AE}$ $\quad\vec{OM} = k\vec{OD}$

$\vec{OD} = \vec{OC} + \vec{CD} = \tilde{c} + \frac{1}{2}\tilde{a}$

$\vec{AE} = \vec{AO} + \vec{OE}$
$= -\tilde{a} + \frac{1}{3}\tilde{c}$

$* \ \vec{OM} = \vec{OA} + \vec{AM}$
$k\vec{OD} = \tilde{a} + h\vec{AE}$
$k(\tilde{c} + \frac{1}{2}\tilde{a}) = \tilde{a} + h(-\tilde{a} + \frac{1}{3}\tilde{c})$
$k\tilde{c} + \frac{k\tilde{a}}{2} = \tilde{a} - h\tilde{a} + \frac{h}{3}\tilde{c}$

$k\tilde{c} - \frac{h}{3}\tilde{c} = \tilde{a} - h\tilde{a} - \frac{k}{2}\tilde{a}$

$\tilde{c}(k - \frac{h}{3}) = \tilde{a}(1 - h - \frac{k}{2})$

ie $k - \frac{h}{3} = 0$ & $1 - h - \frac{k}{2} = 0$

solve simultaneously on CAS

$\left\{\begin{array}{l} k - h/3 = 0 \\ 1 - h - k/2 = 0 \end{array}\right| h, k$

$h = 6/7$
$k = 2/7$

18. HARLEQUIN ⇒ 9 letters

a) 6 letter words

$\binom{9}{6} \times 6!$

$= 60480$

how many have at least 1 vowel?

Vowels = A E U I

Consonants = $\underbrace{\text{H R L Q N}}_{\text{only 5}}$

So 1 has to be vowel

∴ all 6 letter words contain at least 1 vowel

b) PORTHCAWL

$\binom{9}{6} \times 6! = 60480$

Vowel = O A

Consonant = P R T H C W L

at least 1 vowel
= 1 vowel or 2 vowel

$\left\{\binom{2}{1}\binom{7}{5} + \binom{2}{2}\binom{7}{4}\right\} \times 6!$

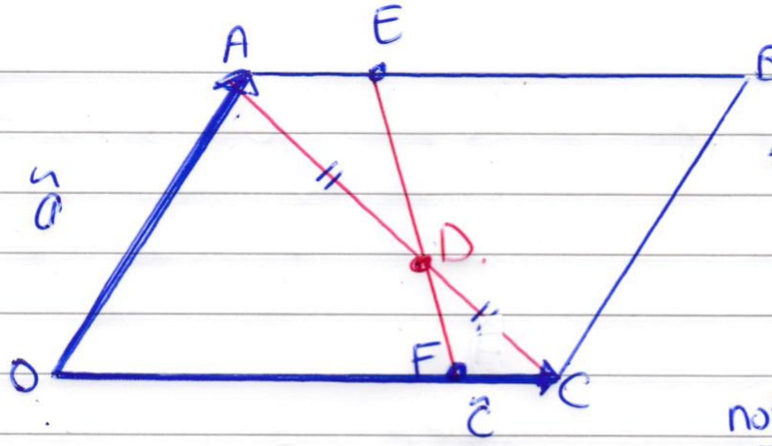
$= 55440$

19. 12 people 2 groups of 6

$\binom{12}{6}\binom{6}{6} \div 2 = 462$

choose a group of 6, then the rest make another group ÷ 2 for double ups

20.



$* \ \overrightarrow{AE} = h\overrightarrow{AB}$

$\overrightarrow{CF} = k\overrightarrow{CO}$

note OABC is a parallelogram

$\overrightarrow{AB} = \overrightarrow{OC} = \tilde{c}$

a) $\overrightarrow{ED} = \overrightarrow{EA}^{*} + \overrightarrow{AD}$ ①

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\tilde{a} + \tilde{c}$

$\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AC}$

$= \frac{1}{2}(-\tilde{a} + \tilde{c})$

$= -\frac{1}{2}\tilde{a} + \frac{1}{2}\tilde{c}$

from ①

$\overrightarrow{ED} = -h\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$

$\overrightarrow{ED} = -h\tilde{c} + -\frac{1}{2}\tilde{a} + \frac{1}{2}\tilde{c}$

$= \frac{1}{2}\tilde{c} - h\tilde{c} - \frac{1}{2}\tilde{a}$ ▲

b) $\overrightarrow{DF} = \overrightarrow{DC} + \overrightarrow{FC}$ ②

$= \frac{1}{2}\overrightarrow{AC} - \overrightarrow{CF}$

$= \frac{1}{2}(-\tilde{a} + \tilde{c}) - k\overrightarrow{CO}$

$= -\frac{1}{2}\tilde{a} + \frac{1}{2}\tilde{c} - k\tilde{c}$

$= \frac{1}{2}\tilde{c} - k\tilde{c} - \frac{1}{2}\tilde{a}$ ■

c) $\overrightarrow{ED} = m\overrightarrow{DF}$

▲ $= m$ ■

ie $\frac{1}{2}\tilde{c} - h\tilde{c} - \frac{1}{2}\tilde{a} = m\left(\frac{1}{2}\tilde{c} - k\tilde{c} - \frac{1}{2}\tilde{a}\right)$

$\frac{1}{2}\tilde{c} - h\tilde{c} - \frac{1}{2}\tilde{a} = \frac{m}{2}\tilde{c} - mk\tilde{c} - \frac{m}{2}\tilde{a}$

$\frac{1}{2}\tilde{c} - h\tilde{c} - \frac{m}{2}\tilde{c} + mk\tilde{c} = \frac{1}{2}\tilde{a} - \frac{m}{2}\tilde{a}$

$\tilde{c}\left(\frac{1}{2} - h - \frac{m}{2} + mk\right) = \tilde{a}\underbrace{\left(\frac{1}{2} - \frac{m}{2}\right)}_{=0}$

ie $\underline{\underline{m = 1}}$

$\therefore \ \overrightarrow{ED} = \overrightarrow{DF}$

ie $\frac{1}{2}\tilde{c} - h\tilde{c} - \frac{1}{2}\tilde{a} = \frac{1}{2}\tilde{c} - k\tilde{c} - \frac{1}{2}\tilde{a}$

$\frac{1}{2}\tilde{c} - h\tilde{c} - \frac{1}{2}\tilde{c} + k\tilde{c} = \frac{1}{2}\tilde{a} - \frac{1}{2}\tilde{a}$

$\tilde{c}\left(\frac{1}{2} - h - \frac{1}{2} + k\right) = 0$

ie $\underbrace{}_{=0}$

$\frac{1}{2} - h - \frac{1}{2} + k = 0$

$-h + k = 0$

$\therefore \ h = k$